Complex surfaces with CAT(0) metrics

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Abstract

We study complex surfaces with locally CAT(0) polyhedral Kähler metrics and construct such metrics on $\mathbb{C}P^2$ with various orbifold structures. In particular, in relation to questions of Gromov and Davis-Moussong we construct such metrics on a compact quotient of the two-dimensional unit complex ball. In the course of the proof of these results we give criteria for Sasakian 3-manifolds to be globally CAT(1). We show further that for certain Kummer coverings of $\mathbb{C}P^2$ of sufficiently high degree their desingularizations are of type $K(\pi, 1)$.

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1 Introduction

In this article we study curvature properties of *polyhedral Kähler manifolds* introduced in [P]. Before recalling the definition of polyhedral Kähler manifolds it is worth to describe two types of examples that are rather familiar. The first class are quotients of complex tori with a flat Kähler metric by a finite group of isometries. The second class are ramified covers of $\mathbb{C}^n$ branched along a collection of complex hyperplanes (these examples are not compact). In both cases we have a complex manifold (smooth or singular) with a flat metric with singularities. Metric singularities of complex co-dimension 1 occur along flat geodesic strata. In the first case the *conical angles* in the normal direction to such strata are $\frac{2\pi}{k}$ while in the second they are $2\pi k$, $k \in \mathbb{N}$. Polyhedral Kähler manifolds have similar structure, but the conical angle at geodesic divisors can be arbitrary.

Now, recall the definition. A *polyhedral Kähler manifold* is a PL manifold M^{2n} with a flat metric with singularities and a compatible parallel complex structure defined outside of metric singularities. The metric should satisfy two conditions. 1) M^{2n} should admit a simplicial decomposition such that the metric restricts to Euclidean metric on each simplex. As a consequence the singularities of the metric represent a union of several $2n - 2$–dimensional simplexes. 2) Every $2n - 2$-dimensional simplex contained in the singular locus of the metric is holomorphic with respect to the complex structure on each top dimensional simplex containing it. We use abbreviation PK to denote polyhedral Kähler manifolds and metrics on them.

It was proven in [P] that 4-dimensional PK manifolds are smooth complex surfaces and the metric singularities form a union of complex curves on these surfaces. For the purposes of this article we widen the class of PK manifolds in the 4-dimensional case and consider PK metrics defined on complex surfaces (of complex dimension two) with isolated complex singularities.

We give criteria on the singularities of PK metrics implying that the metric is locally CAT(0) (see Definition 2.2), and use them to construct several families of locally CAT(0) orbifold metrics on $\mathbb{C}P^2$. In particular we produce

a locally CAT(0) polyhedral metric on a compact complex hyperbolic 2-ball quotient (see Section 4.3). This is related to questions raised by Gromov ([Gr2] , Section 7.A.IV, p. 180), and Davis with Moussong (see discussion in [DM] Question 2, Section 7.3). Further we construct several families of complex surfaces of type $K(\pi, 1)$.

1.1   Criteria for being locally CAT(0)

A metric on a polyhedral Kähler manifold is flat outside of singularities, so to check that it is locally CAT(0) we need to study metric singularities. On a 4-dimensional PK manifold metric singularities occur in real codimension 2 (along complex curves) and in codimension 4 (where these curves meet). The first case is easy, the metric is locally a direct product of a 2-dimensional cone with $\mathbb{R}^2$; it is locally CAT(0) if an only if the 2-cone has conical angle larger than 2π. The real task central for this article is to understand the nature of locally CAT(0) condition at metric singularities of codimension 4.

For every point x of a PK 4-manifold we denote by $C_K^4(x)$ its *tangent cone*, i.e., the cone that has a neighborhood of its origin isometric to a neighborhood of x. By $S_K^3(x)$ we denote the unit sphere in the tangent cone. A metric on a PK surface is locally CAT(0) if and only if for every point x the sphere $S_K^3(x)$ is CAT(1). By Corollary 2.6 this happen if and only if any closed geodesics in $S_K^3(x)$ is not shorter than 2π, and the conical angle at each complex curve from the singular locus is greater than 2π.

Recall [P] that for any point x on a PK 4-manifold the cone $C_K^4(x)$ admits an isometric holomorphic action by $\mathbb{R}^1$, given by the field $ir\frac{\partial}{\partial r}$ (here $r\frac{\partial}{\partial r}$ is the usual dilatation field that exists on every polyhedral cone). If $C_K^4(x)$ is not a direct product of two cones of angles $2\pi\beta_1$ and $2\pi\beta_2$ with $\frac{\beta_1}{\beta_2} \notin \mathbb{Q}$, then the isometric action of $\mathbb{R}^1$ factors through an S^1-action. In this article we concentrate on the cones where this S^1 action on $S_K^3(x)$ is free, and we call such cones *regular spherical cones*. It was shown in [P] (Theorem 1.7) that for a regular spherical cone there exists a canonical (up to scaling) biholomorphism $C_K^4(x) \rightarrow \mathbb{C}^2$ that sends the singular locus of $C_K^4(x)$ to a collection of lines passing through the origin of $\mathbb{C}^2$. It will also be important for us to consider ramified covers of regular spherical cones with branching at the lines where the metric has singularities. These are polyhedral cones not biholomorphic to $\mathbb{C}^2$ and we call them *regular* if the natural action of S^1 on them is free.

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The next two theorems provide two main classes of locally CAT(0) metric singularities that appear in this article.

Theorem 1.1 *Let S_K^3 be the unit sphere of a regular spherical PK cone C_K^4. If the quotient 2-sphere S_K^3/S^1 is CAT(4) then the sphere S_K^3 is CAT(1) and consequently the cone C_K^4 is locally CAT(0).*

Theorem 1.1 is related to results obtained [CD]. Namely, Theorem 9.1 from [CD] gives a *necessary* and *sufficient* criterion for polyhedral metrics on $\mathbb{R}^4$ whose quotient by a complex reflection group is isometric to the flat $\mathbb{R}^4$, to be CAT(0). Theorem 1.1 extends the *sufficiency* conditions from [CD] to the whole class of spherical PK cones. We prove as well a smooth analogue of Theorem 1.1 for *Sasakian* metrics on S^3, Theorem 3.10.

Next theorem explains that locally CAT(0) cones can be constructed by taking ramified covers of non-negatively curved cones.

Theorem 1.2 *Let S_K^3 be the unit sphere of a regular spherical PK cone C_K^4. Let $s_1, ..., s_k$, $k \geq 3$ be the singular circles of the metric on S_K^3. Suppose that the conical angles along the circles $s_1, ..., s_k$ are less than 2π. Then for any sufficiently large integers $n_1, ..., n_k > 0$ there exists a 3-manifold M^3 of type CAT(1) with an isometric action of a finite group G such that the quotient M/G is isometric to S_K^3 and has branching of order n_i along s_i.*

1.2 Constructions of complex surfaces and orbifolds

Orbifold CAT(0) **metrics on** $\mathbb{C}P^2$. In order to construct orbifold CAT(0) metrics on $\mathbb{C}P^2$ we use line arrangements denoted by $A_1(6)$, $A_1(7)$ and $A_3^0(3)$ in [G] and [H] (for the definition see Section 4.3). These arrangements satisfy two important properties. First, they are singular loci of PK metrics on $\mathbb{C}P^2$ with conical angles less than 2π ([P]). Second, the multiple points of these arrangements have multiplicity at most three. Each of these arrangements gives rise to an infinite series of locally CAT(0) orbifold structures on $\mathbb{C}P^2$, where to each line we associate a certain orbifold multiplicity d so that the orbifold stabilizer at non-multiple point of the line is $\mathbb{Z}_d$. In Section 4.2 we explain further how the orbifold structure is defined at multiple points.

Theorem 1.3 *Let $(L_1, ..., L_n)$ be one of the arrangements $A_1(6)$, $A_1(7)$ or $A_3^0(3)$. Associate integers $d_i > 1$ to lines L_i in such a way that at every triple point of the arrangement incoming numbers d_i, d_j, d_k satisfy $\frac{1}{d_i} + \frac{1}{d_j} + \frac{1}{d_k} > 1$.*

Then for each orbifold structure $(L_1, d_1; ...; L_n, d_n)$ *on* $\mathbb{C}P^2$ *there exits a PK metric that is locally* CAT(0) *with respect to the orbifold structure.*

Orbifolds in Theorem 1.3 have contractible universal covers. In several cases the universal cover is bi-holomorphic to $\mathbb{C}^2$, polidisk, or unit complex ball. Some of these cases have appeared in [U], and all such orbifods are representable as finite quotients of smooth complex surfaces. However for majority of orbifolds in Theorem 1.3 the universal cover is not of one of these three types. One can expect that such orbifolds are finite quotients of negatively curved surfaces, analogous to surfaces of Mostow-Siu [MS]. See also Remark 4.6 where we explain how one can find an orbifold CAT(-1) metric in the case of arrangement $A_3^0(3)$.

Aspherical complex surfaces. Using Kummer covers of $\mathbb{C}P^2$ and Theorem 1.2 we produce several series of aspherical complex surfaces of type $K(\pi, 1)$. Recall that Kummer covers are associated to line arrangements $(L_1, ..., L_k)$ on $\mathbb{C}P^2$ and an integer $n > 0$. One considers the extension of the field of rational functions on $\mathbb{C}P^2$ by functions $(\frac{L_i}{L_j})^{\frac{1}{n}}$. This extension defines a Galois cover $S(n, L_1, ..., L_k) \to \mathbb{C}P^2$ of degree n^{k-1}.

Theorem 1.4 *Consider a PK metric on* $\mathbb{C}P^2$ *that has conical angles less than* 2π *at the line arrangement* $L_1, ..., L_k$. *There exists* N *such that for every* $n \geq N$ *the blow up of* $S(n, L_1, ..., L_k)$ *at its complex singularities is an aspherical complex surface.*

The fundamental groups of these surfaces are quite non-trivial infinite quotients of the fundamental group of the complement to arrangement (L_i) in $\mathbb{C}P^2$. The proof of the theorem relies on the following general statement.

Theorem 1.5 *Let* S *be complex surface with isolated complex singularities. If* S *admits a locally* CAT(0) *metric then any blow up of* S *that does not contain rational exceptional curves is of type* $K(\pi, 1)$.

We note finally that Kummer covers were extensively studied starting form the work of Hirzebruch [H], who used them to construct complex ball quotients that are blow ups of $\mathbb{C}P^2$ at several points.

2   Generalities about $\mathrm{CAT}(\kappa)$ spaces and proof of Theorem 1.5

In this section we mainly recall definitions and theorems about $\mathrm{CAT}(\kappa)$ spaces. The material is borrowed from Chapter 10 of [GH] and Chapter 4 of [Gr1] unless a different source is specified.

Denote by M_κ^k a complete simply connected manifold of constant curvature κ, i.e., the hyperbolic k-space for $\kappa < 0$, the flat $\mathbb{R}^k$ for $\kappa = 0$ and the k-sphere (of radius $\frac{1}{\sqrt{\kappa}}$) for $\kappa > 0$.

Definition 2.1 Let X be a complete geodesic space. A *triangle* Δ in X is a union of three geodesic segments that join three points of X. The *comparison triangle* of Δ in M_κ^2 is the triangle $\tilde{\Delta}$ with the edges of the same length as Δ. We say that Δ satisfies the $\mathrm{CAT}(\kappa)$ inequality if for all $x, y \subset \Delta$ and the corresponding points $\tilde{x}, \tilde{y} \subset \tilde{\Delta}$ the following inequality holds:

$$d(x, y) \leq d(\tilde{x}, \tilde{y}).$$

A space X is called a $\mathrm{CAT}(\kappa)$ space if every triangle in X of perimeter less than $\frac{2\pi}{\sqrt{\kappa}}$ satisfies the $\mathrm{CAT}(\kappa)$ inequality (if $\kappa \leq 0$ the inequality should be satisfied for all triangles).

Definition 2.2 A space X has curvature $K_X \leq \kappa$ if for every point $p \in X$ there is a neighborhood U_p such that every triangle in U_p of perimeter less than $\frac{2\pi}{\sqrt{\kappa}}$ satisfies the $\mathrm{CAT}(\kappa)$ inequality (if $\kappa \leq 0$ we consider all triangles). If $K_X \leq \kappa$ we also say that X is *locally* $\mathrm{CAT}(\kappa)$.

Theorem 2.3 *Suppose that $K_X \leq \kappa$ and every closed geodesic in X has length at least $\frac{2\pi}{\sqrt{\kappa}}$. Then X is a $\mathrm{CAT}(\kappa)$ space.*

Theorem 2.4 *A simply connected geodesic space X with $K_X \leq 0$ is contractible, every two points on it are joined by a unique geodesic.*

Proof of Theorem 1.5. Let $\widetilde{S}$ be the universal cover of S, and $\widetilde{S}_\circ$ the blow up of $\widetilde{S}$ that covers the chosen blow up of S. To prove the theorem it is sufficient to show that $\widetilde{S}_\circ$ is $K(\pi, 1)$. Since $\widetilde{S}_\circ$ is a blow up of $\widetilde{S}$ in a discrete subset of complex singularities, any homotopy of a sphere in $\widetilde{S}_\circ$ can touch only finite number of exceptional curves. Hence it is enough to prove that the blow up of $\widetilde{S}$ at any finite subset of complex singularities, say $x_1, ..., x_n$ is $K(\pi, 1)$.

Fix a point x of $\widetilde{S}$. By Theorem 2.4 for any $1 \leq i \leq n$ there exists a unique geodesic γ_i on $\widetilde{S}$ that joins x and x_i. The union of the geodesics γ_i forms a tree on $\widetilde{S}$, we denote this tree by T and denote by T_ε the ε-neighborhood of T. Since $\widetilde{S}$ is CAT(0) every geodesic ray starting at x intersects the boundary ∂T_ε exactly at one point. Indeed, in any CAT(0) space for any geodesic ray $r(t)$ and a geodesic segment $s(t)$ sharing common vertex $x = r(0) = s(0)$ the distance from $r(t)$ to the segment s is a function increasing with t. It follows that T_ε is contractible, and as well $\widetilde{S}$ can be contracted on T_ε by a homotopy identical on T_ε and preserving geodesics rays starting at x.

Notice finally that by the assumption of the theorem the blow up of T_ε at $x_1, ..., x_n$ is homotopic to a bouquet of n spaces of type $K(\pi, 1)$. Indeed, the blow up of T_ε is homotopic to the bouquet of ε-neighborhoods $U_\varepsilon(x_i)$ of x_i blown up at points x_i. At the same time every $U_\varepsilon(x_i)$ blown up at x_i can be contracted on its exceptional curve, which is a union of several curves of genus higher than 0 with several points identified. Such spaces are $K(\pi, 1)$. $\square$

Polyhedra of curvature $\kappa \leq K$. We use the terminology from [Gr1]. A *straight simplex* in M_κ^k is a compact intersection of $k+1$ half-spaces in M_κ^k in generic position. Let X be a finite-dimensional simplicial complex with a metric such that each simplex is isometric to a straight simplex in M_κ^k for some k. Such a space X is called (M, κ) space. To each k-simplex Δ on X we can associate its *link* L_Δ. This is an $(M, 1)$ space that can be defined as the set of unit vectors orthogonal to Δ at an interior point. Here to each simplex Δ' containing Δ corresponds a simplex in $L(\Delta)$ of curvature 1 and of dimension $dim(\Delta') - k - 1$. The following theorem is stated, for example on page 2 in [Bo].

Theorem 2.5 *Let X be a (M, κ_0) space. Then $K_X \leq \kappa$ if and only $\kappa_0 \leq \kappa$*

and for any Δ in X the link L_Δ does not contain a closed geodesic of length less than 2π.

Of course, a PK manifold is an $(M, 0)$ space. In order to apply Theorem 2.5 to a 4-dimensional PK cone we need to verify the CAT(1) condition on the link at its origin and as well at all complex curves forming the singular locus of the cone. For such curves the link is just the circle of length equal to the conical angle at the curve. This is summarized in the following corollary:

Corollary 2.6 *A PK cone C_K^4 is locally CAT(0) if and only if the conical angle at any curve in its singular locus is greater than 2π and closed geodesics on its unitary sphere S_K^3 have length at least 2π.*

The following result on CAT(κ) spaces can be found in [AKP].

Theorem 2.7 *Let X_n be a sequence of CAT(κ) spaces that converges in Gromov-Hausdorff topology to X. Then X is also a CAT(κ) space.*

Finally we quote a comparison theorem from [AB] on curves with bounded geodesic curvature on CAT(κ) spaces. The definition of geodesic curvature for curves in CAT(κ) spaces is introduced on page 70 in [AB], it is called *arch-chord curvature*. We don't reproduce the definition here since it is a bit technical, while the comparison theorem will be applied at worst to piecewise smooth curves on real surfaces with conical points. By a *loop* in a CAT(κ) space we mean a rectifiable curve γ that admits a parameterization $\phi[0, 1] \to \gamma$ with $\phi(0) = \phi(1)$. The following statement is a special case of Theorem 1.1 [AB] that treats as well curves with $\phi(0) \neq \phi(1)$.

Theorem 2.8 *Let X_κ be CAT(κ) space and let γ a loop in it of geodesic curvature at most k (the curvature need not be defined at the base point of γ). Then γ is no shorter than the complete circle of curvature k on M_κ^2.*

3   CAT(1) property of unit spheres of PK cones

The goal of this section is to prove Theorems 1.1 and 1.2. Let us outline briefly the strategy of the proofs. In both theorems we need to show that on the unit sphere S_K^3 of a certain regular PK cone all closed geodesics have length at least 2π. The first step is to project geodesics in S_K^3 to the quotient S_K^3/S^1 and notice that the projection is a curve of constant curvature on the

complement to conical points of S_K^3/S^1 (Lemma 3.1). One can show then that in the case when S_K^3 is topologically S^3 and conical points on S_K^3/S^1 have angles greater than 2π, every closed geodesic on S_K^3 projects to a curve with a self-intersection on S_K^3/S^1 (Proposition 3.7). This observation joined with Theorem 2.8 settles Theorem 1.1. Theorem 1.2 is deduced from its 2-dimensional analogue (Corollary 3.15) joined with Theorem 2.8.

In addition to Theorem 1.1 we prove its smooth version (Theorem 3.10) for Sasakian 3-manifolds. Both proofs follow the same path and for the sake of exposition we give them simultaneously, giving all the details for the smooth case and explaining how to alter them for the original case of Theorem 1.1. In principle, using carefully Theorem 2.7 one could also show that the statements of Theorem 1.1 and its smooth version 3.10 are equivalent. Indeed every singular metric on S^3 from 1.1 can be Gromov-Hausdorff approximated by Sasakian metrics from 3.10 and vice versa. We chose not to do this to avoid technical issues that arise in this approach.

3.1   Geodesics of unit spheres of PK singularities

Lemma 3.1 *Let γ be a geodesic on the unit sphere S_K^3 of a regular PK cone. Then the angle $\angle\gamma'$ between γ and the S^1-fibration is constant. The lengths of γ and it projection $p(\gamma)$ to S_K^3/S^1 are related by the formula $l(\gamma)\sin(\angle\gamma') = l(p(\gamma))$. Moreover $p(\gamma)$ is a piecewise smooth curve of constant curvature $2\frac{\cos(\angle\gamma')}{\sin(\angle\gamma')}$ that can have singularities only at conical points of S_K^3/S^1.*

2) Let $x \in p(\gamma)$ be a conical point on S_K^3/S^1. Then locally $p(\gamma)$ divides a neighborhood of x into two sectors with angles at x greater or equal to π.

Proof. All these statements are obvious in the case when S_K^3 is the unit sphere of the flat $\mathbb{C}^2$, with the standard S^1 action defining the Hopf fibration on S_K^3. In this case the quotient S_K^3/S^1 is the sphere of curvature 4.

1) In the general case any closed geodesic γ on the unit sphere S_K^3 of a regular PK cone is composed of smooth geodesic segments that join singular fibers of the S^1-fibration. The function $\angle\gamma'$ is constant on each smooth segment of γ because the S^1-fibration on the complement to singularities in S_K^3 is locally isometric to the Hopf fibration on the standard sphere S^3.

Let us show that the value of $\angle\gamma'$ is the same on all smooth segments of γ. Denote by $T^2 = S^1 \times \gamma$ the torus immersed in S_K^3 that is the union of S^1 fibers intersecting γ. The restriction metric on T^2 is flat, γ is embedded in T^2 as a geodesic and so it intersects the flat foliation given by S^1-fibers

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under constant angle. Hence we have $\angle \gamma' = \text{const}$ and the further claims of the lemma hold since they hold for the standard unit sphere.

2) If one of two sectors of $p(\gamma)$ at x had angle smaller than π one would be able to smoothen γ in a neighborhood of $p^{-1}x$ to make it shorter, which is impossible since γ is locally minimizing.

$\square$

3.2   Geodesics on Sasakian 3-manifolds

In this section we recall the definition of regular *Sasakian* 3-manifolds and study geodesics on them. Let $M^3 \to M^2$ be an S^1-bundle over a real surface M^2.

Definition 3.2 A metric g on M^3 is called regular *Sasakian* if it satisfies the following conditions:

1) There is an isometric action of S^1 on M^3 that preserves the fibers

2) Let ω_0 be the 1-form of norm 1 with zero distribution orthogonal to the S^1-fibers. Then $d\omega_0$ equals $-2dS$, where dS is the area form on M^3/S^1.

Regular Sasakian manifolds are special case of Sasakian [BG], that don't necessarily admit a structure of S^1-fibration. The simplest example of a Sasakian 3-manifold is the unit sphere where the S^1-fibration is the Hopf fibration. Unit spheres or PK cones can be considered as examples of Sasakian manifolds with singular metric.

The following is the main statement of this section, I would like to thank Robert Bryant for the help with its proof.

Proposition 3.3 *Let γ be a geodesic on a regular Sasakian manifold M^3. The angle between γ and the S^1-fibration is constant and the projection of γ to M^2 is a smooth curve of constant curvature equal to $2\omega_0$.*

Proof. Let $\omega_0, \omega_1, \omega_2$ be S^1-invariant 1-forms on M^3 forming an orthogonal base of T^*M^3, and such that the restriction of ω_1 and ω_2 on S^1-fibers are zero. Let e_0, e_1, e_2 be the dual vector fields on M^3. Then $[e_0, e_1] = [e_0, e_2] = 0$, and $[e_1, e_2] = 2e_0$. Plugging this in the formula for the exterior derivative of 1-forms $d\omega(X, Y) = L_X\omega(Y) - L_Y\omega(X) - \omega([X, Y])$ we get:

$$d\omega_0 = -2\omega_1 \wedge \omega_2, \quad d\omega_1 = -\theta_{12} \wedge \omega_2, \quad d\omega_2 = \theta_{12} \wedge \omega_1, \tag{1}$$

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where θ_{12} is a uniquely defined 1-form, that is a (point-wise) linear combination of w_1 and w_2.

Let p_0, p_1, p_2 be the coordinates on fibers of T^*M^3 dual to $\omega_0, \omega_1, \omega_2$. Then the canonical 1-form α on T^*M^3 is the following:

$$\alpha = p_0\omega_0 + p_1\omega_1 + p_2\omega_2. \tag{2}$$

Using formula (1) and (2) we get the following expression for the symplectic form on T^*M^3:

$$d\alpha = dp_0 \wedge \omega_0 + (dp_1 + p_2\theta_{12} - p_0\omega_2) \wedge \omega_1 + (dp_2 - p_1\theta_{12} + p_0\omega_1) \wedge \omega_2. \tag{3}$$

Denote by X the geodesic flow on T^*M^3. From Lee formula it follows

$$i(X)d\alpha = -\alpha = -p_0dp_0 - p_1dp_1 - p_2dp_2. \tag{4}$$

Using $\omega_i X = p_i$ we deduce form (3) and (4) the following system of equations

$$dp_0 X = 0, \quad (dp_1 + p_2\theta_{12} - p_0\omega_2)X = 0, \quad (dp_2 - p_1\theta_{12} + p_0\omega_1)X = 0.$$

Here the first equation states that any geodesic on M^3 has fixed angle with the S^1 fibration, namely the quantity $p_0(X)$ is constant along the trajectories of the geodesic flow. On the over hand, if we fix $p_0(X) = c$ the next two equations can be interpreted as the equations for the twisted geodesic (magnetic) flow on $T^*(M^3/S^1)$, whose trajectories project to curves of geodesic curvature $2c$ on M^3/S^1. In particular by choosing $c = 0$ we get the geodesic flow. To go from M^3 to M^3/S^1 we use the S^1-invariance of all equations. □

Lemma 3.4 *Consider a regular Sasakian metric on S^3 and let $S^1 \hookrightarrow S^3 \to S^2$ be the corresponding fibration. Then the equally holds $l(S^1) = 2\mathrm{Area}S^2$. The same relation holds for the unit sphere of a regular spherical PK cone.*

Proof. In the Sasakian case applying Stokes formula to a disk in S^3 whose boundary is a fiber S^1 and using the definition of Sasakian metric we get $\int_{S^1} \omega_0 = - \int d\omega_0 = 2\mathrm{Area}(S^2)$. The second statement is a special case of Theorem 1.9 [P]. □

Finally we give one standard proposition.

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Proposition 3.5 *The sectional curvature of a regular Sasakian 3-manifold M^3 is bounded from above by 1 if $K(M^3/S^1) \leq 4$.*

Proof. This can be found, for example in Section 7 [BG]. The sectional curvature of each 2-plane containing the vertical direction equals 1, and the curvature of a horizontal 2-plane equals $K(M^3/S^1) - 3$ (this can also be deduced from O'Neill's formula).

$\square$

3.3 S^3 is CAT(1) if S^2 is CAT(4)

In this subsection we prove Theorem 1.1 and its smooth analogue.

Proposition 3.6 *1) Let γ be a closed geodesic in a regular Sasakian manifold M^3, whose projection to M^3/S^1 bounds a topological disk Ω_γ. Stokes formula on M^3 applied to (ω_0, γ) and, Gauss-Bonnet applied to γ on M^3/S^1 give us:*

$$\int_\gamma w_0 \equiv -2Area(\Omega_\gamma) \mod l(S^1), \tag{5}$$

$$\int_\gamma w_0 = \frac{1}{2}(2\pi - \int_{\Omega_\gamma} K \, dS). \tag{6}$$

2) Suitably interpreted Equalities (5) and (6) hold as well when M^3 is a unit sphere of a regular polyhedral Kähler cone.

Proof. 1) To prove Equality (5) consider a section s of the S^1 bundle over Ω_γ. Let ∂s be the restriction of this section over $\partial\Omega_\gamma$. Using the definition of Sasakian metric and applying Stokes formula we get

$$-2Area(\Omega_\gamma) = \int_S dw_0 = \int_{\partial S} w_0 \equiv \int_\gamma w_0 \mod l(S^1).$$

Equality (6) follows from Proposition 3.3 together with Gauss-Bonnet formula.

2) In the case when M^3 is the unit sphere of a PK cone, Equation (5) does not require any interpretation and is proved as above, one just needs to use Lemma 3.1 instead of Proposition 3.3. To interpret correctly (6) we first notice that the curvature K on M^3/S^1 is the sum of the constant 4 with delta-functions, supported at conical points. A conical point x of conical

angle $2\pi\alpha(x)$ supports the delta function $\delta(x) = 2\pi(1 - \alpha(x))\delta$. In order to interpret correctly the integral $\int_{\Omega_\gamma} K \, dS$ we should specify the contribution to it of conical points lying on the boundary of Ω_γ. Let y be such a point and let $\pi\alpha_{in}(y)$ be the angle between two arcs of γ measured inside of Ω_γ, and $\pi\alpha_{out}(y)$ be the angle measured in the complement $(M^3/S^1) \setminus \Omega_\gamma$. Of course, the sum $\pi\alpha_{in}(y) + \pi\alpha_{out}(y)$ equals the total conical angle $2\pi\alpha(y)$ at y, and $\alpha_{in} \geq 1$, $\alpha_{out} \geq 1$ since γ is a geodesic. We split the delta function $\delta(y)$ as a sum of $\delta_{in}(y) = \pi(1 - \alpha_{in}(y))\delta$ and $\delta_{out}(y) = \pi(1 - \alpha_{out}(y))\delta$. Assigning to each conical point y on the boundary of Ω_γ the delta function $\delta_{in}(y)$ one gets the correct version of formula (6).

$\square$

Proposition 3.7 *Consider a regular Sasakian metric on S^3 and let $S^3 \to S^2$ be the corresponding S^1-fibration. Suppose that $K(S^2) \leq 4$. Then for any closed geodesic γ on S^3 its projection to S^2 has a point of self-intersection.*

Proof. Suppose by contradiction, that $p(\gamma)$ has no self-intersection on S^2. Denote by Ω_γ a disk on S^2 bounded by γ. Subtracting the first equation of 3.6 from the second we get

$$0 \equiv \pi + \int_{\Omega_\gamma} (2 - \frac{K}{2}) \, dS \;\; mod \;\; l(S^1). \tag{7}$$

One the other hand using $K \leq 4$ with $\int_{S^2} \frac{K}{2} \, dS = 2\pi$, we get

$$\pi \leq \pi + \int_{\Omega_\gamma} (2 - \frac{K}{2}) \, dS = -\pi + 2S(\Omega_\gamma) + \int_{S^2 \setminus \Omega_\gamma} \frac{K}{2} dS < 2Area(S^2) - \pi, \tag{8}$$

these inequalities contradict Equation (7) since $l(S^1) = 2Area(S^2)$ by 3.4.

$\square$

Remark 3.8 The statement of Proposition 3.7 holds as well if S^3 is isometric to the unit sphere of a regular PK cone, such that the conical angles of the quotient S_K^3/S^1 are greater than 2π. The proof goes as above, one only needs to justify the equality inside Equation (8). To do this we follow the recipe of Proposition 3.6 2), namely for each conical point x on $p(\gamma)$ we split the contribution of the curvature δ-function $\delta(x)$ as $\delta_{in}(x) + \delta_{out}(x)$, with δ_{in} contributing to Ω_γ, while δ_{out} contributing to $S^2 \setminus \Omega_\gamma$.

Let ϕ be a closed curve on a real surface, $\phi : S^1 \to M^2$. A piece $\phi[t_1, t_2]$ of the curve ϕ is a *loop* if $\phi(t_1) = \phi(t_2)$. The following lemma is trivial.

Lemma 3.9 *A closed curve with self-intersections on a real surface has at least two loops $\phi[t_1, t_2]$ and $\phi[t_3, t_4]$ that have no overlap, i.e., $(t_1, t_2) \cap (t_3, t_4) = \emptyset$.*

Let us now formulate and proof the smooth analog of Theorem 1.1.

Theorem 3.10 *Consider a regular Sasakian metric on S^3 such that the quotient S^3/S^1 is CAT(4). Then S^3 is CAT(1).*

Proof. By Proposition 3.5 the sectional curvature of S^3 is bounded from above by 1, so to prove that S^3 is CAT(1) we need to show that the length of any closed geodesic γ on S^3 is at least 2π.

By Proposition 3.7 the projection $p(\gamma)$ of γ to S^3/S^1 has at least one self-intersection. This means that $p(\gamma)$ contains at least two loops. By Theorem 2.8, each loop is no shorter than the complete circle of the same geodesic curvature of the sphere of curvature 4. Hence by Proposition 3.3 the length of a piece of γ that projects to a loop is at least π. We deduce $l(\gamma) \geq 2\pi$.

$\square$

Proof of Theorem 1.1. This proof repeats the proof of 3.10 with the following changes. One needs to adjust Proposition 3.7 according to Remark 3.8. In order to apply (comparison) Theorem 2.8 to a loop of $p(\gamma)$ we need to show that the geodesic curvature of $p(\gamma)$ at conical points of the surface (where $p(\gamma)$ is not smooth) is not larger than at a smooth point of the curve. This follows directly from definition of geodesic curvature on page 70 of [AB]. Finally, instead of Proposition 3.3 we use Lemma 3.1.

$\square$

Remark 3.11 It would be interesting to show that these results hold in higher dimensions, for example to prove that a regular Sasakian metric on S^{2n+1} is CAT(1) if the metric on the quotient $\mathbb{C}P^n \cong S^{2n+1}/S^1$ is CAT(4).

3.4 CAT(4) **covers of positively curved 2-spheres**

In this subsection we explain that a curvature 4 sphere with conical singularities of angles *less* than 2π admits ramified covers such that the metric pulled back to the cover is CAT(4) (Corollary 3.15). First, we will need a result showing that conical points on such a sphere are distributed "evenly".

Proposition 3.12 *Let S^2 be a sphere with a metric of curvature 4 with conical points $x_1, ..., x_k$, $k > 2$ of angles less than 2π. Then for any point $x \in S^2$ there exists x_i such that $d(x, x_i) < \frac{\pi}{4}$ (x_i coincides with x if x is conical itself).*

The proof of the proposition uses a theorem of Toponogov and a lemma.

Theorem 3.13 (Toponogov, [To].) *Let g be a metric on S^2 with Gaussian curvature $K \geq 4$. Then any simple closed geodesic on S^2 has length at most π. If one of such geodesics has length π, then the metric g is of curvature 4.*

Lemma 3.14 *For a sphere S^2 of curvature 4 with conical points and a smooth point x on it the following statements hold:*

1) Suppose that there is a geodesic loop γ of length at most $\frac{\pi}{2}$ based at x. Then there is a conical point on S^2 of distance at most $\frac{\pi}{4}$ from x.

2) Suppose that the distance from x to all conical points is larger than d, and $d > \frac{\pi}{4}$. Then S^2 contains an isometric copy of a spherical disc of radius d with the center at x.

Proof. 1) The geodesic loop γ cuts a neighborhood of x into two sectors and the angle of one of the sectors is at most π, denote this angle by α. Let ABC be a geodesic triangle on the standard sphere of curvature 4 such the length of AB is $|\gamma|$ and the angles at A and B are $\frac{\alpha}{2}$. Glue by isometry CA with CB identifying B with C so that we get a figure that we call a *cup*. The *cup* has one conical point in its interior (the point corresponding to C), and by construction its boundary is a geodesic loop of length $|\gamma|$ with angle α at its base point. Notice that in the triangle ABC the distance from any point to the set $\{A, B\}$ is at most $\frac{\pi}{4}$ (ABC is contained in a equilateral triangle with side $\frac{\pi}{4}$). Hence the distance from any point of the *cup* to the angle on its boundary is at most $\frac{\pi}{4}$.

Consider now a local isometry from a neighborhood of the boundary of the *cup* to a neighborhood of γ. Suppose by contradiction that all conical points on S^2 are on distance at least $\frac{\pi}{4}$ to x. Then this local isometry can be extended to the local isometry from the *cup* minus its conical point. Hence it extends by continuity to the conical point, whose image is a conical point on S^2 on the distance at most $\frac{\pi}{4}$ from x.

2) Since there are no conical points on S^2 on distance less than $\frac{\pi}{4}$ from x, it follows from 1) that the radius of injectivity at x is at least $\frac{\pi}{4}$. Hence S^2 contains an isometric copy of the discs of radius $\frac{\pi}{4}$ centered at x. Consider

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now the family of disks centered at x of radius growing from $\frac{\pi}{4}$ to d. The boundaries of these disks are smooth (they don't contain conical points) and concave, hence they can not acquire a self-intersection as the radius grows. Therefore they are all embedded in the sphere.

$\square$

Proof of Proposition 3.12. Suppose first by contradiction that there is a point on S^2 on distance more than $\frac{\pi}{4}$ from each conical point. Then by Lemma 3.14 2) the sphere contains an isometric copy of a radius $\frac{\pi}{4}$ disk. Its boundary is a simple closed geodesic of length π. This contradicts Toponogov's theorem, since we can smoothen the metric near conical points so that the curvature is still at least 4 and the geodesic is kept intact.

To finish the proof we need to rule out the case when the distance from a point x on S^2 to all conical points is a least $\frac{\pi}{4}$ and for some i $d(x, x_i) = \frac{\pi}{4}$. By arguments similar to those used in the proof of Lemma 3.14 one can check in this case that S^2 has exactly two conical points and x lies on the closed geodesic of length less than π that splits S^2 into two equal parts. But by the assumptions the number of conical points on S^2 is at least 3.

$\square$

Corollary 3.15 *Let S^2 be a sphere with a metric of curvature 4 with conical points $x_1, ..., x_k$, $k > 2$ of angles less than 2π. Then we have:*

1) There exists a geodesic triangulation of S^2 by convex triangles with vertices at $x_1, ..., x_k$.

2) For sufficiently large multiplicities $n_1, ..., n_k$ the universal orbifold cover of the orbifold $(S^2, x_1, n_1, ..., x_k, n_k)$ is a CAT(4) *space.*

3) For any cover from 2) there exists $C > 0$ such that simple closed curves γ on the cover satisfy the isoperimetric inequality $Area(\gamma) < C \cdot l(\gamma)^2$.

Proof of Corollary 3.15.1) This proof is almost identical to the proof of Proposition 3.1 in [Th] claiming the existence of a triangulation for a sphere with a flat metric and with conical points of angles less than 2π. We will just recall the main idea.

One shows first that there exists a canonical decomposition of S^2 into a collection of convex polygons whose edges are geodesics joining conical points of S^2. To construct the polygons one considers all (finite number) maps from round disks to S^2 that are locally isometric on the interior and such that at least 3 conical points are contained in the image of the boundary of the disk. The polygon is obtained as the convex hull of all conical points on the disk

boundary. Once polygons are constructed each of them can be decomposed into triangles. Notice that we use here Proposition 3.12 to assure the all considered disks are of radius less than $\frac{\pi}{4}$.

□

The proof of 3.15 2) is slightly less elementary than one would want it to be but at least it avoids calculations. We will use a couple of results of Bowditch and a simple lemma.

Theorem 3.16 (Bowditch [Bo].) *In a complete locally compact locally CAT(1) space a closed geodesic cannot be freely homotoped to a point through rectifiable curves of length strictly less than 2π.*

Lemma 3.17 *Any piecewise smooth closed curve on a compact constant curvature surface with conical points can be freely homotoped to a closed geodesic or a point via rectifiable curves of decreasing length.*

This last lemma is a very special case of Theorem 3.1.6 from [Bo].

Lemma 3.18 *For a convex spherical triangle Δ_4 of curvature 4 and its comparison triangle Δ_κ of curvature $\kappa \leq 4$ there exists a bi-Lipschitz map $\Delta_4 \to \Delta_\kappa$ that is an isometry on the boundary. Moreover the bi-Lipschitz constant is bounded by a continuous function $c(\Delta_4, \kappa)$ with $c(\Delta_4, 4) = 1$.*

Proof. Let O_4 and O_κ be the centers of circles inscribed in Δ_4 and Δ_κ correspondingly. Consider the map sending isometrically the boundary of Δ_4 to the boundary of Δ_κ and sending geodesic segments through O_4 to geodesic segments through O_κ so that the metric restricted to each geodesic segment is multiplied by a constant. A calculation shows that this map is bi-Lipschtiz and the bi-Lipschtiz constant tends to 1 as κ tends to 4.

□

Proof of Corollary 3.15.2) Consider the triangulation of S^2 by convex triangles given by 3.12 1). Replacing each convex triangle by a comparison triangle of curvature $0 \leq \kappa \leq 4$ we get a family of spheres S_κ^2 of curvature κ with conical points, S_4^2 being the original sphere. The cone angles of S_κ^2 decrease with κ, and we denote by α_{i0} the cone angle of S_0^2 at x_i. The sphere S_0^2 is of curvature 0 and for $n_1, ..., n_k$ such that $n_i \alpha_{i0} \geq 2\pi$ the orbifold universal cover $\widetilde{S}_0^2$ of (S_0^2, n_i, x_i) is CAT(0). It follows that $\widetilde{S}_0^2$ does not contain closed geodesics. We will deduce from this that $\widetilde{S}_4^2$ does not contain closed geodesics of length π or less, and so it is CAT(4).

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Suppose by contradiction that on $\widetilde{S}_4^2$ there is a geodesic of γ length $\pi - \varepsilon$. Denote by T the subset of $[0, 4]$ of κ such that $\widetilde{S}_\kappa^2$ contains a geodesic of length at most $\pi - \varepsilon$. By standard arguments T is closed and we will show that it is also open. This will imply that $T = [0, 4]$ and will lead to a contradiction since $0 \notin T$. To show that T is open we need to prove that for each $\kappa \in T$ a neighborhood of κ belongs to T, we will do it for $\kappa = 4$, for other points the reasoning is identical.

Chose δ so that for each triangle Δ_4 from the convex triangulation of $\widetilde{S}_4^2$ we have $c(\Delta_4, \kappa) < \sqrt{\frac{\pi}{\pi - \varepsilon}}$ for $\kappa \in [4 - \delta, 4]$. Let us show that for $\kappa \in [4 - \delta, 4]$ $\widetilde{S}_\kappa^2$ contains a geodesic of length less than $\pi - \varepsilon$. By Lemma 3.18 for $\kappa \geq 4 - \delta$ there exists a bi-Lipschitz map F_κ from $\widetilde{S}_4^2$ to $\widetilde{S}_\kappa^2$ with bi-Lipschitz constant less than $\sqrt{\frac{\pi}{\pi - \varepsilon}}$. Notice that the curve $F_\kappa(\gamma)$ can not be contracted to a point on $\widetilde{S}_\kappa^2$ by a length decreasing homotopy. Indeed, during such a homotopy the length of the curve would be at most $l(F_\kappa(\gamma)) \leq \sqrt{\pi(\pi - \varepsilon)}$. Hence the pull back of such a homotopy to S_4^2 would contract γ to a point keeping its length less than π, which contradicts Theorem 3.16.

Denote by γ' the piecewise geodesic curve on $\widetilde{S}_\kappa^2$ obtained from $F_\kappa(\gamma)$ by replacing each interval of $F_\kappa(\gamma)$ with the ends at the boundary of the triangulation by a geodesic segment. The curve $F_\kappa(\gamma)$ can be deformed to γ' by a length decreasing deformation. By Lemma 3.17 γ' can be homotoped further to a shorter geodesic γ''. Finally, it follows from the $\mathrm{CAT}(\kappa)$ inequality (Definition 2.1) that $l(\gamma') \leq l(\gamma)$. Hence as promised we constructed on $\widetilde{S}_\kappa^2$ a closed geodesic γ'' with $l(\gamma'') < \pi - \varepsilon$. This finishes the proof that $\widetilde{S}_4^2$ is $\mathrm{CAT}(4)$.

$\square$

Proof of Corollary 3.15.3) In order to prove that curves on $\widetilde{S}_4^2$ satisfy the isoperimetric inequality recall that there is a bi-Lipschitz map from $\widetilde{S}_4^2$ to $\widetilde{S}_0^2$, and the last space is $\mathrm{CAT}(0)$, hence curves on it satisfy the desired isoperimetric inequality.

$\square$

3.5   Proof of Theorem 1.2 and its refinements

Let C_K^4 be a regular spherical PK cone. According to Theorem 1.7 [P] the cone is canonically (up to a scale) biholomorphic to $\mathbb{C}^2$ and its singular locus is defined by a collection of linear equations $L_1 = 0, ..., L_k = 0$ in $\mathbb{C}^2$. Moreover, the multiplication of $\mathbb{C}^2$ by unitary complex numbers generates

a free isometric S^1-action on C_K^4. For $n > 1$ consider the ramified (Galois) covering of C_K^4 of degree n^k corresponding to the extension of the field of rational functions on $\mathbb{C}^2$ by functions $L_1^{\frac{1}{n}}, ..., L_k^{\frac{1}{n}}$. The corresponding PK cone that covers C_K^4 is denoted by $(C_K^4)^{\frac{1}{n}}$, the unit sphere of this cone is denoted by $(S_K^3)^{\frac{1}{n}}$. One can check that the isometric S^1-action on C_K^4 generated a *free* isometric action on $(C_K^4)^{\frac{1}{n}}$ (and as well on its unit sphere $(S_K^3)^{\frac{1}{n}}$).

Proposition 3.19 *Let C_K^4 be a positively curved regular spherical PK cone. For sufficiently large n every closed geodesic γ in the unit sphere $(S_K^3)^{\frac{1}{n}}$ of the cone $(C_K^4)^{\frac{1}{n}}$ with contractible projection $p(\gamma)$ on $(S_K^3)^{\frac{1}{n}}/S^1$ has length at least 2π.*

   Proof. Denote by M_n^2 the universal cover of $(S_K^3)^{\frac{1}{n}}/S^1$ and by M_n^3 the corresponding cover of $(S_K^3)^{\frac{1}{n}}$. Then any geodesic γ in $(S_K^3)^{\frac{1}{n}}$ with contractible projection on $(S_K^3)^{\frac{1}{n}}/S^1$ admits a lift to M_n^3. So it is sufficient to show that for n large enough any closed geodesic γ' in M_n^3 is longer than 2π.
   Note that the cover M_n^2 is isomorphic to the orbifold universal cover of the sphere S_K^3/S^1 with orbifold structure of multiplicity n at all its conical points. Hence by Corollary 3.15.2) M_n^2 is CAT(4) for n large enough. If $p(\gamma')$ has a self-intersection then it has at least two loops each contributing at least π to the length of γ and so $l(\gamma') \geq 2\pi$ (see Theorem 2.8 and the proof of Theorem 3.10). Recall now Equation (5) and (6)

$$\int_{\gamma'} w_0 \equiv -2Area(\Omega_{\gamma'}) \mod l(S^1), \quad \int_{\gamma'} w_0 = \frac{1}{2}(2\pi - \int_{\Omega_{\gamma'}} K \ \mathrm{dS}).$$

Using $K \leq 4$ we get

$$\int_{\gamma'} w_0 = \frac{1}{2}(2\pi - \int_{\Omega_{\gamma'}} K \ \mathrm{dS}) \geq \pi - \int_{\Omega_{\gamma'}} 2\mathrm{dS} > -2Area(\Omega_{\gamma'}). \qquad (9)$$

   Suppose now that $l(\gamma') < 2\pi$. Then the isoperimetric inequality (Corollary 3.15 3)) tells us $Area(\Omega_{\gamma'}) < C(4\pi^2)$. At the same time we can chose n such that $l(S^1) > 2\pi + 2C(4\pi^2)$. For such n Equation (5) with Inequality (9) imply $\int_{\gamma'} w_0 > 2\pi$. Hence $l(\gamma') > 2\pi$ and we get a contradiction.
$\square$

   Proposition 3.19 should be compared with the following lemma.

Lemma 3.20 *Suppose that C_K^4 is not a direct product of two 2-dimensional cones, and let α_{min} be the minimal conical angle at the singular locus of C_K^4. If $\alpha_{min} \cdot [\frac{n}{2}] \geq \pi$ then $(C_K^4)^{\frac{1}{n}}$ is not locally CAT(0).*

Proof. Indeed, if C_K^4 is not a direct product then any two conical points x and y on the sphere S_K^3/S^1 can be joined by a geodesic segment $[xy]$ with $l([xy]) < \frac{\pi}{2}$. Consider in S_K^3 a geodesic segment $[x'y']$ that joins singular Hopf fibers over x and y and projects to $[xy]$. This segment is orthogonal to the S^1 fibers and has the same length as $[xy]$. Consider now on the complement to singular circles in S_K^3 the following path contained in an ε-neighborhood of $[x'y']$. The path starts in the middle of $[x'y']$, goes to the singular circle containing x', enlaces it $[\frac{n}{2}]$ times, goes to y', enlaces the singular circle $[\frac{n}{2}]$ times and comes back to the middle of $[x'y']$, after this the path repeats itself again, but enlaces the circles in the opposite direction. If follows from the construction of $(C_K^4)^{\frac{1}{n}}$ that this path lifts to a closed path on $(S_K^3)^{\frac{1}{n}}$. Moreover in the case when the conical angles at x and y satisfy $\alpha(x)[\frac{n}{2}] > \pi$ and $\alpha(y)[\frac{n}{2}] > \pi$ there is a closed geodesic in $(S_K^3)^{\frac{1}{n}}$ composed of 4 preimages of $[x'y']$, contained in the ε-neighborhood of the lifted path.

$\square$

Remark 3.21 Notice that the cone $(C_K^4)^{\frac{1}{n}}$ is not CAT(0) if $\alpha_{min} \cdot n < 2\pi$, so it is reasonable to guess that it can be CAT(0) only if C_K^4 is a direct product of two 2-cones. Lemma 3.20 also explains why the degree n^2 cover $\mathbb{C}^2 \to \mathbb{C}^2$ with order n branching at two non-orthogonal lines is not CAT(0).

The following Proposition provides an explicit construction for Theorem 1.2.

Proposition 3.22 *For every cone $(C_K^4)^{\frac{1}{n}}$ satisfying the conditions of Proposition 3.19 there exists a cone $\widetilde{C}_K^4$ satisfying the following properties:*

1) There is an action of a finite group G on $\widetilde{C}_K^4$ such that $(C_K^4(x_i))^{\frac{1}{n}} = \widetilde{C}_K^4/G$; the action of G is free outside of the origin of $\widetilde{C}_K^4$.

2) Closed geodesics in the unit sphere $\widetilde{S}_K^3$ are longer than 2π.

3) The map of $\widetilde{C}_K^4$ blown up at the origin to $(C_K^4)^{\frac{1}{n}}$ blown up at the origin is a non-ramified covering.

Proof. Consider the Riemann surface $M_n = (S_K^3)^{\frac{1}{n}}/S^1$. Since $\pi_1(M_n)$ is a residually finite group (see [He] for an elementary proof) there exists

a Galois covering $\widehat{M}_n \to M_n$ with finite Galois group G such that all non-contractible geodesics on $\widehat{M}_n$ are longer than 2π.

Denote by $(C_K^4)_\circ^{\frac{1}{n}}$ the cone $(C_K^4)^{\frac{1}{n}}$ blown up at the origin. This is the total space of a certain line bundle over M_n and so $\pi_1((C_K^4)_\circ^{\frac{1}{n}}) \cong \pi_1(M_n)$. Take the Galois covering of $(C_K^4)_\circ^{\frac{1}{n}}$ with the Galois group G. Then the cone $\widetilde{C_K^4}$ obtained by contraction of the exceptional curve on the constructed covering, satisfies the properties 1), 2), 3). Indeed, 2) is verified because any geodesic γ on $\widetilde{S_K^3}$ with contractible $p(\gamma)$ is longer than 2π by Proposition 3.19; if $p(\gamma)$ is non-contractible, then by construction $l(\gamma) \geq l(p(\gamma)) > 2\pi$. The properties 1) and 3) follows automatically from the construction of $\widetilde{C_K^4}$.

$\square$

Proof of Theorem 1.2. Chose n such that the cone $(C_K^4)^{\frac{1}{n}}$ satisfies the conditions of Proposition 3.19 and take its cover provided by Proposition 3.22. The unit sphere of the obtained cone is the desired 3-manifold.

$\square$

Remark 3.23 In Theorem 1.2 one can let conical angles tend to 2π. As a result we get the statement that for each collection $L_1, ..., L_k$ of complex lines in a flat $\mathbb{C}^2$ going trough 0 one can construct a locally CAT(0) ramified cover of $\mathbb{C}^2$ with branchings at L_i, provided the following condition holds. For the points $p_1, ..., p_k$ on the projectivization $\mathbb{C}P^1$ of $\mathbb{C}^2$ corresponding to $L_1, ..., L_k$ there is no geodesic S^1 in $\mathbb{C}P^1$ such that $p_1, ..., p_k$ are contained in one connected component of $\mathbb{C}P^1 \setminus S^1$.

4 Construction of complex surfaces

4.1 Proof of Theorem 1.4

In order to prove Theorem 1.4 we need to use the notion of *rigid orbispaces* due to Haefliger, defined in [GH], chapter 11. This is a natural generalization of the notion of orbifolds to the category of locally compact topological spaces. Rigid orbispace is defined by replacing the model for the orbifold charts by a locally compact space with a rigid action of a finite group, i.e. one for which points with trivial isotropy are dense. The precise definition can be found in [GH] and [CD]. We say that a geodesic metric space X with a structure of a rigid orbispace is *non-positively curved* if the metric induced

on each model chart is locally CAT(0).

The following theorem is proven in Chapter 11 of [GH].

Theorem 4.1 *Any non-positively curved rigid orbispace X admits a universal cover, i.e., there exists a simply connected topological space with an action of a group G such that $Y/G = X$, $\mathrm{stab}(y)$ is finite for any $y \in Y$, and the orbispace structure induced on X by the quotient coincides with the original one.*

Proof of Theorem 1.4. Let $x_1, ..., x_m$ be the multiple points of the arrangement. Let $C_K^4(x_1), ..., C_K^4(x_m)$ be the tangent cones to $\mathbb{C}P^2$ at the points $x_1, ..., x_m$. Chose such N' that for any $n \geq N'$ the cones $(C_K^4(x_1))^{\frac{1}{n}}, ..., (C_K^4(x_m))^{\frac{1}{n}}$ satisfy the condition of Proposition 3.19. Put

$$N = \max\left(N', \max_j \frac{1}{\beta_j}\right).$$

Pullback the metric from $\mathbb{C}P^2$ to the complex surface $S(n, L_j)$. If $n \geq N$ the conical angles at singular curves on $S(n, L_j)$ are greater than 2π, and so the metric has non-positive curvature on the complement to the pre-images of the points $x_1, ..., x_m$. At the same time each point $\widetilde{x}_i$ in the preimage of x_i has a neighborhood isometric to a neighborhood of the origin in the cone $(C_K^4(x_i))^{\frac{1}{n}}$. Using Proposition 3.22 we can represent each cone $(C_K^4(x_i))^{\frac{1}{n}}$ as a quotient of a locally CAT(0) cone by a finite group (where only the origin has a nontrivial stabiliser). This defines on the complex surface $S(n, L_j)$ a structure of a rigid non-positively curved orbispace.

By Theorem 4.1 the complex surface $S(n, L_j)$ with the constructed rigid orbispace structure admits a universal orbispace cover $\widetilde{S}$. The pullback metric on $\widetilde{S}$ has non-positive curvature, so by Theorem 1.5 the complex surface $\widetilde{S}_\circ$ obtained from $\widetilde{S}$ by the blowup of its complex singularities is of the type $K(\pi, 1)$. At the same time by condition 3) of Proposition 3.22 the complex surface $\widetilde{S}_\circ$ is a non-ramified cover of the blow up of $S(n, L_j)$ at its complex singularities. Hence $S(n, L_j)$ is of the type $K(\pi, 1)$ as well.

$\square$

4.2   Orbifold structures on complex surfaces

Consider a smooth complex surface S with a divisor $D = \sum_{i=1}^n D_i$ such that the divisors D_i are smooth, irreducible and pairwise transversal. Suppose

we want to introduce on S the structure of an orbifold so that for points of D_i that are not multiple in D the local fundamental group is $\mathbb{Z}_{b_i}$, $b_i > 1$. This imposes conditions on the multiple points of D and on the numbers b_i summarised in the next proposition (see for example [Y] Section 11, or [U]).

Proposition 4.2 *The complex surface (S, D) admits an orbifold structure with local fundamental group $\mathbb{Z}_{b_i}$ at generic points of D_i if all multiple points of D are at most triple, and at each triple point $\frac{1}{b_j} + \frac{1}{b_k} + \frac{1}{b_l} > 1$ for the incoming multiplicities. The order of the local fundamental group at the triple point is $4(\frac{1}{b_j} + \frac{1}{b_k} + \frac{1}{b_l} - 1)^{-2}$.*

This proposition relies on the list of Shephard and Todd of complex reflection groups acting on $\mathbb{C}^2$. For each triple point of the divisor infinitesimally one has to get an action of G on $\mathbb{C}^2$ such that $\mathbb{C}^2/G \cong \mathbb{C}^2$ and moreover the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/G$ is a branched cover that ramifies along three lines in $\mathbb{C}^2/G \cong \mathbb{C}^2$. For each triple of branching multiplicities p, q, r satisfying $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ there is a unique group in the list of Shephard and Todd. The group is called $G(2s, 2, 2)$ in the case $(p = s, q = r = 2)$, and has numbers (7), (11), and (19) for triples $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$ correspondingly. These groups act on the projectivisation $\mathbb{C}P^1$ of $\mathbb{C}^2$ and the action factors through the action of a dihedral group for $G(2s, 2, 2)$, and tetrahedral, octahedral, and dodecahedral groups in the other 3 cases.

Orbifold Chern numbers and Miyoka-Yau. Recall that according to a result of Miyaoka and Yau for a complex surface S of general type the inequality on Chern numbers holds $c_1^2(S) \leq 3c_2(S)$ and the equality is attained if and only if the universal cover of S is the unit ball $|z_1|^2 + |z_2|^2 < 1$. The analogous statement for orbifolds is contained in [KNS]. We will only state a very special case of this theorem when $S \cong \mathbb{C}P^2$ and the divisor $D = D_1 + ... + D_n$ satisfies the conditions of Proposition 4.2. Denote by d_i the degree of D_i and for each singular point p of D denote by $\beta_D(p)$ the order of the local group at p. Then the Chern numbers of the orbifold $(\mathbb{C}P^2, D, b)$ are defined as follows.

$$c_1^2(\mathbb{C}P^2, D, b) = (-3 + \sum_{1 \leq i \leq n} d_i(1 - b_i^{-1}))^2 \tag{10}$$

$$e(\mathbb{C}P^2, D, b) = 3 - \sum_{1 \leq i \leq n} (1 - b_i^{-1})e(D_i \setminus sing(D)) - \sum_{sing(D)} (1 - \beta_D(p)^{-1}) \tag{11}$$

The following theorem is a special case of a theorem from [KNS].

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Theorem 4.3 *Let* $(\mathbb{C}P^2, D, b)$ *be an orbifold that is not a global quotient of* $\mathbb{C}P^2$. *Then* $c_1^2(\mathbb{C}P^2, D, b) \leq 3e(\mathbb{C}P^2, D, b)$, *the equality holding if and only if* $(\mathbb{C}P^2, D, b)$ *is uniformized by the complex ball.*

Corollary 4.4 *The orbifold* $(\mathbb{C}P^2, A_3^0(3), 2)$ *is uniformized by the complex ball.*

Proof. The arrangement $A_3^0(3)$ consist of 9 lines each of which intersect others in 4 points. Each multiple point p is triple with $\beta_D(p) = 16$, and there are 12 of them. Applying equations (10) and (11) we get

$$c_1^2(\mathbb{C}P^2, A_3^0(3), 2) = \frac{9}{4} = 3e(\mathbb{C}P^2, A_3^0(3), 2),$$

and since $(\mathbb{C}P^2, A_3^0(3), 2)$ is not uniformized by $\mathbb{C}P^2$ the proof is finished by Theorem 4.3. $\square$

4.3 Proof of Theorem 1.3

Let us first give a description of the arrangements $A_1(6)$, $A_1(7)$, and $A_3^0(3)$.

1) The *complete quadrilateral* $A_1(6)$ consists of 6 lines on $\mathbb{C}P^2$ that pass through 4 generic points. The corresponding PK metric on $\mathbb{C}P^2$ has conical angles π at all lines.

2) $A_1(7)$ consists of lines $x = \pm z$, $y = \pm z$, $x = \pm y$, and $z = 0$. The metric has angles π at lines $x = \pm z$, $y = \pm z$, and angles $\frac{4\pi}{3}$ at the rest.

3) $A_3^0(3)$ is the arrangement of 9 lines given by the equations $x^3 - y^3 = 0$, $y^3 - z^3 = 0$ and $z^3 - y^3 = 0$ with angles $\frac{4\pi}{3}$ at all of them.

Theorem 1.3 will be deduced from Theorem 1.1 and a special case of a lemma from [Gr1]. Call a convex triangle on S^2 of curvature 4 *large* if the distance from each vertex to the opposite side is a least $\frac{\pi}{4}$.

Lemma 4.5 *(4.2.E, Remark b, [Gr1].)* *Consider a real surface of curvature 4 with conical singularities admitting a triangulation by large triangles of valence at least four. If any two conical points are joined by at most one edge and any cycle consisting of 3 edges borders a face of the triangulation then the surface is* CAT(4).

Sketch of a proof. Let γ be a closed geodesic on the surface. If γ is composed entirely of the edges of the triangulation then it should contain at least four edges, hence $l(\gamma) \geq \pi$. Otherwise γ should intersect an edge e

of the triangulation. Let Δ_1 and Δ_2 be the triangles adjacent to e and let A_1, A_2 be the vertices of Δ_1 and Δ_2 opposite to e. Let $L(A_1)$ and $L(A_2)$ be the links of A_1 and A_2 consisting of all triangles containing A_1 and A_2 correspondingly. From the condition of the lemma it follows that the interiors of $L(A_1)$ and $L(A_2)$ don't intersect. The main observation is that if a closed geodesic intersects a link of a vertex on the surface then the length of the intersection is at least $\frac{\pi}{2}$.

$\square$

Proof of Theorem 1.3. For the arrangements $A_1(6)$, $A_1(7)$ and $A_3^0(3)$ in [P] were constructed PK metrics on $\mathbb{C}P^2$ such that all conical angles along the lines of the arrangements are at least π. In the case of $A_1(6)$ the conical angles at all six lines are π, for $A_1(7)$ four angles are π and three angles are $\frac{4\pi}{3}$, for $A_3^0(3)$ the angles at all nine lines are $\frac{4\pi}{3}$. Since the orbifold structure along each line is non trivial the orbifold metrics have non-positive curvature in the complement of the multiple points of the arrangements.

Let x be a triple point of $A_1(6)$, $A_1(7)$, or $A_3^0(3)$. In order to prove that the orbifold has non-positive curvature at x we need to show that the orbifold cover of the unit sphere $S_K^3(x)$ is CAT(1). By construction this cover is the unit sphere of a regular spherical PK cone, and so by Theorem 1.1 it is sufficient to prove that the orbifold cover of $S_K^3(x)/S^1$ is CAT(4).

The sphere $S_K^3(x)/S^1$ is a 2-sphere of curvature 4 with three conical points whose angles are as follows: (π, π, π) for $A_1(6)$, $(\frac{4\pi}{3}, \frac{4\pi}{3}, \pi)$ for $A_1(7)$, and $(\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3})$ for $A_3^0(3)$. In each case the sphere is obtained by gluing two copies of a large triangle. Hence the corresponding orbifold covers of $S_K^3(x)/S^1$ are triangulated by large triangles and it is straightforward to check that these triangulations satisfy the conditions of Lemma 4.5.

$\square$

Remark 4.6 The statement of Theorem 1.3 can be strengthen in the case of arrangement $A_3^0(3)$. It follows from [CHL] that the Fubini-Studi metric on $\mathbb{C}P^2$ admits a one parameter deformation in the class of metrics of constant curvature with singularities at the arrangement $A_3^0(3)$; the family is parameterized by the conical angle at the lines of $A_3^0(3)$. Both the flat metric that we used and the ball quotient metric (from Corollary 4.4) belong to this family, corresponding conical angles being $\frac{4\pi}{3}$ and π. Using the methods of the present article one can show that each metric from the family with conical angles in the interval $(\pi, \frac{4\pi}{3})$ is locally CAT(K), $K < 0$ with respect to any orbifold structure from Theorem 1.3.

Locally CAT(0) **polyhedral metrics on hyperbolic manifolds.** Theorem 1.3 can be used to construct a locally CAT(0) polyhedral metric on a compact complex surface that is a smooth quotient of the unit two-dimensional complex ball. Namely, we should apply Theorem 1.3 to the arrangement $A_3^0(3)$ with choices $d_1 = ... = d_9 = 2$. By Corollary 4.4 the orbifold universal cover of $\mathbb{C}P^2$ is the unit complex ball B^2. So the orbifold fundamental group is a co-compact lattice in $U(2,1)$, and taking a quotient of B^2 by its torsion free finite index subgroup we get an example.

By [CDM] it is known that every real hyperbolic manifold (of sectional curvature -1) admits a locally CAT(0) polyhedral metric, but the existence of such a metric on a complex hyperbolic manifold is rather unexpected (see the discussion on page 45 in [DM]). In [Gr2] Gromov introduces a class of *polyhedral groups with $K < 0$*, that are fundamental groups of locally CAT(0) polyhedrons, satisfying additional property of *non-flatness at each face* (page 176). He makes a conjecture (Section 7.A.IV, page 180) that for every group of this class any homomorphism to a co-compact lattice in $U(n,1)$ should factor through a map to π_1 of a Riemann surface. If the polyhedral metrics that we construct on ball quotients admit a polyhedral decomposition, such that each face is *not flat* in the sense of [Gr2], one would get a counterexample to the above conjecture. It should not be difficult to check if we actually get a counterexample but we have not yet done that.

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